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Book Review

"Navier–Stokes Equations: An Introduction with Applications". G. Łukaszewicz, and P. Kalita, Advances in Mechanics and Mathematics, vol. 34, Springer International Publishing, 2016;
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The history of Navier–Stokes Equations (NSE) begins with the 1822 paper of C.L.M.H. Navier (Ann. Chim. Phys. 19, 234–245) who derived equations for homogeneous incompressible fluids from a molecular point of view. The continuum derivation of the NSE is due to J.C. Saint-Venant (1843) and G.G. Stokes (Trans. Cambridge Philos. Soc. 1845, 8, 287–319). The NSE are generally treated as the universal basis of fluid mechanics, no matter how complicated and unpredictable the behavior of its solutions may be. These equations are, nowadays, at the foundations of many branches of applied sciences, including atmospheric physics, oceanography, geology and geophysics, aerodynamics, civil and hydro-engineering, and biology and medicine.

This book is devoted to the mathematical theory of the NSE for incompressible fluids. The basic, classical, and non-classical tools are included, and part of the material comes from original articles published by the authors. The book contains sixteen chapters, beginning with one that gives a terse overview of the subject and describes the layout of the remainder of the book in more detail. It also offers more than 80 exercises that accompany almost every chapter. The proofs of some theorems are presented as sets of exercises. Moreover, some exercises contain the supplementary information on the problem under consideration. Furthermore, each chapter is concluded with a brief discussion of the literature and essential remarks concerning, among others, the priority results.

Chapter 2, *Equations of Classical Hydrodynamics*, deals with some basic notions and facts of the general theory of continuous media. The first section is devoted to the kinematics of motion and serves as a vehicle for introducing the concept of transformation map. The Reynolds transport theorem, which is fundamental in continuum mechanics, is presented next. This theorem is used to derive the field equations of classical hydrodynamics from the conservation laws of mass, momentum, and energy. The simplest assumption concerning the stress tensor, leading to linear Stokesian fluids, is utilized. The presentation of Navier–Stokes equations then follows, and particular attention is focused on the vorticity transport equation, emphasizing the role of vorticity and vortex dynamics together with a brief discussion of concepts from irreversible thermodynamics of homogeneous fluid. The use of the similarity theory to describe the fluid motion is also discussed, whereas in penultimate section the three exact solutions are demonstrated. These include the flow due to an impulsively moved plain boundary, the Poiseuille flow, and the Couette flow. The last section contains some remarks related to the essence of hydrodynamic modeling.

Chapter 3, *Mathematical Preliminaries*, contains necessary mathematical technicalities, which are essential for a proper understanding of more qualitative or more concrete questions. In the first section, the Lax–Milgram lemma is formulated and proved using the Galerkin method, together with corollary (Riesz–Fréchet theorem for separable spaces). Subsequently, the fixed point theorems are discussed, and

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the elements of the theory of function spaces are presented. Particular emphasis is given to Sobolev spaces, distributions and distributional derivatives, some embedding theorems, and useful inequalities (Poincaré, Ladyzhenskaya, and Hardy inequalities). Some proofs are given in case if they are relevant to the development of the subject, or do not seem to be widely known. The basic properties of evolution spaces, including evolution triples, are also specified. Gronwall-type inequalities are then introduced and briefly analyzed in the following section. The remaining sections contain a terse description of the Clarke subdifferential (or generalized gradient). This notion is the generalization of Gâteaux derivative to functionals which are only locally Lipschitz. Some results regarding the properties of the Clarke subdifferential and multifunctions, together with the examples, are demonstrated.

Chapters 4–6 deal with stationary problems and concentrate on some results related to the existence, uniqueness, and regularity of solutions. Chapter 4, *Stationary Solutions of the Navier–Stokes Equations*, is devoted to the stationary NSE in a bounded three-dimensional domain with periodic or homogeneous boundary conditions. First, in order to obtain a well-posed boundary value problem, the suitable function spaces are introduced. Next, the existence theorem is proved, using the Galerkin method and fixed point theorems, as well as the uniqueness of solutions for large viscosity coefficients with respect to mass forces. The alternative proofs of the existence of a solution, using the other topological methods, are also outlined.

Chapter 5, *Stationary Solutions of the Navier–Stokes Equations with Friction*, addresses the three-dimensional stationary Navier–Stokes problem with multivalued friction law boundary conditions on a part of the domain boundary. Two existence theorems are proved for the analyzed problem. The first case concerns the linear growth condition on the friction multifunction, and the second one deals with the case of power growth assumption. The proofs of theorems are obtained using the Kakutani–Fan–Glicksberg fixed point method and some cut-off techniques, respectively.

Chapter 6, *Stationary Flows in Narrow Films and the Reynolds Equation*, focuses on the mathematical aspects of the lubrication problems. The famous problem from the theory of lubrication, namely, the Stokes flow in a thin three-dimensional domain is considered. After recalling the weak formulation of the problem, the key estimates and the necessary inequalities are introduced and proved. By making the appropriate scaling of the problem, the convergence theorem for the rescaled functions is formulated and established. Next, the limit variational inequality is obtained, and the strong convergence of the velocity fields is proved. An important discussion on the choice of suitable function spaces, in order to enable the passing of the limit, is performed. Consequently, the Reynolds-like equation and the limit boundary conditions are derived, and the uniqueness of solutions of the limit problem is obtained. Finally, some comments on lubrication problems, and their related mathematical and technical subtleties, are also presented.

Chapters 7–10 survey some results about the non-stationary autonomous NSE in two-dimensional domains. Chapter 7, *Autonomous Two-Dimensional Navier–Stokes Equations*, introduces some mathematical apparatus and techniques of dealing with the non-stationary NSE. The weak formulation of the initial boundary value problem for the NSE is given first. The periodic or homogeneous Dirichlet boundary conditions are considered. Next, the existence of a unique global in time weak solution is proved. After introduction the basic concepts on dynamical systems and attractors, the existence of the global attractor for the semigroup associated with the two-dimensional NSE with periodic boundary conditions is proved. The structure of the attractor in some special cases of external forces is also discussed. At last, the direct and inverse transfers of energy and enstrophy within appropriate ranges of length scales are elucidated.

Chapter 8, *Invariant Measures and Statistical Solutions*, concerns the statistical approach to the NSE, that is, deals with measures defined on the space of solutions to the above equations. The concept of invariant measure is introduced, and the definition of the stationary statistical solution of the two-dimensional Navier–Stokes system is given. The proof of the existence of a stationary statistical solution for such system is performed.

Chapter 9, *Global Attractors and a Lubrication Problem*, is dedicated to the two-dimensional boundary-driven shear flow related to lubrication phenomena. At the beginning, some elements of fractal analysis are reviewed. The existing approaches that can be used to prove the finite dimensionality of the global attractor are thoroughly explained. In order to formulate the problem, the background method (Hopf construction) is used. The existence of the unique weak solution of the problem is showed, based on an energy inequality, the Galerkin approximations, and the compactness method. Finally, the estimation the global attractor dimension using the estimate of the time-averaged energy dissipation rate and a Lieb–Thirring-like inequality is achieved.

Chapter 10, *Exponential Attractors in Contact Problems*, treats the problem of time asymptotics for a class of two-dimensional turbulent boundary-driven flows. First of all, the necessary results in the theory of exponential attractors in the autonomous evolutionary case are summarized. Two examples of contact problems are formulated and examined: the planar shear flow with the Tresca friction condition and the planar shear flow with generalized Tresca-type friction law, where the friction coefficient can depend on the tangential slip rate. The first problem is governed by a variational inequality, while the second problem leads to a differential inclusion, where the multivalued term has the form of the Clarke subdifferential. It is proved that in both cases the global attractors exist, and they have finite fractal dimensions. Also, there exists a larger object which contains the global attractor, attracts the trajectories at a fast rate (exponentially fast in time), is still finite dimensional, and is more robust under perturbations. This fact is also proved.

Chapters 11–13 refer to the non-autonomous Navier–Stokes problems. Chapter 13, *Non-autonomous Navier–Stokes Equations and Pullback Attractors*, deals with the time asymptotics of solutions of the two-dimensional NSE. Two properties of the equations in a bounded domain, concerning the existence of determining modes and nodes, are discussed and proved. Subsequently, the notion of asymptotically compact non-autonomous dynamical system is introduced, and basic results referring to the existence of a minimal pullback attractor under assumptions of asymptotic compactness and existence of a family of absorbing sets are recapitulated. Finally, the application of these results in the study of the mathematical properties of general solutions of the two-dimensional NSE in an unbounded domain is presented.

Chapter 12, *Pullback Attractors and Statistical Solutions*, describes the construction of invariant measures and statistical solutions for the non-autonomous NSE in bounded and some unbounded domains. Basic notions and known facts from the theory of pullback attractors in the context of the initial and boundary value problem for the NSE are presented first. After the formulation, the result about the existence of the pullback attractor for the Navier–Stokes problem, the construction of a family of time-averaged probability measures in the phase space of the flow, and their relation to the pullback attractor are provided. It is shown that such family of probability measures satisfies the Liouville equation for a suitable family of generalized moments and the appropriate energy equation. As application, the relation between time-dependent and stationary statistical solutions is established. At last, the statistical solutions of the Navier–Stokes equations in the case of an unbounded domain, where the initial condition is any continuous mapping contained in the domain of attraction of the pullback attractor, are considered.

Chapter 13, *Pullback Attractors and Shear Flows*, pertains to the class of non-autonomous, two-dimensional turbulent boundary-driven flows. After some preliminaries, a weak formulation of the considered problem is given. The proof of the existence and uniqueness of global in time solutions, based on energy inequality, the Galerkin approximations, and the compactness method, is the theme of the next section. Using these results, the existence of the pullback attractor together with the estimate of its fractal dimension is established.

Chapters 14–16 are dedicated to global in time solutions of the NSE. Chapter 14, *Trajectory Attractors and Feedback Boundary Control in Contact Problems*, investigates the two-dimensional nonstationary incompressible fluid shear flows with nonmonotone and multivalued leak boundary conditions on a part of the boundary of the flow domain. This issue is inspired and motivated by recent trends in mechanics connected with the feedback control problems for Newtonian and non-Newtonian fluid flows in domains with semipermeable walls and membranes. After the formulation of the initial boundary value problem, a proof of the existence of solutions of considered problem, which satisfy the energy inequality, is executed. Lastly, the existence of a trajectory attractor and a weak global attractor for the analyzed problem is proved.

Chapter 15, *Evolutionary Systems and the Navier–Stokes Equations*, contains both recent and the latest results of the research devoted to investigation of the three-dimensional non-stationary NSE with the multivalued frictional boundary condition. The formalism of evolutionary systems developed for studying dynamical systems without uniqueness is outlined first. Next, the weak formulation of the considered problem is presented, and, using the Galerkin method, the existence of the weak solution for the problem in the Leray–Hopf sense is proved. The existence of a trajectory attractor and a weak global attractor for the analyzed problem is also established.

Chapter 16, *Attractors for Multivalued Processes in Contact Problems*, deals with the abstract evolution problems in the context of multivalued non-autonomous infinite-dimensional dynamical systems theory. Some elements of the theory of pullback attractors for multivalued processes are presented, including sufficient conditions for the existence of pullback attractors. Then, these techniques are used to study a two-dimensional incompressible fluid flow with a general form of multivalued frictional contact conditions. Only the proof of the attractor existence is demonstrated, because due to non-uniqueness the question of its fractal dimension is an open issue.

This book is very well written, but for the physically oriented researcher it is not easy to read because, owing to the nature of the subject, it assumes the reader's familiarity with several branches of mathematics: calculus, functional analysis, theory of PDEs, dynamical systems theory, and elements of the non-smooth analysis. For the relatively gentle introduction to the mathematical fluid dynamics, I additionally recommend the *Mathematical Tools for the Study of the Incompressible Navier–Stokes Equations and Related Models* by F. Boyer and P. Fabrie (Springer, 2013), and *Initial-Boundary Value Problems and the Navier–Stokes Equations* by H–O. Kreiss and J. Lorenz (SIAM, 2004).

This book will be definitely useful for those who already know the subject and who are interested in mathematical aspects of hydrodynamics. But the following question remains notoriously open: are the Navier–Stokes equations able to provide a rigorous description of the phenomenon of turbulence?